Boundary and global Schauder estimates

Brian Krummel

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Next we want to prove a local Schauder estimate at the boundary of a domain. First we will consider the special case where the domain is locally an upper-half-space. We shall use coordinates $x = (x_1, x_2, \dots, x_n)$ on $\mathbb{R}^n$ and let $x' = (x_1, x_2, \dots, x_{n-1})$. Let $\mathbb{R}^n_+ = \{x : x_n > 0\}$, $\mathbb{R}^n_- = \{x : x_n < 0\}$, and $S = \{x : x_n = 0\}$. Let $B^+_R = B_R(0) \cap \{x : x_n \geq 0\}$ for all $R > 0$ and $S_R = B_R \cap \{x : x_n = 0\}$.

Lemma 1. *Let $\mu \in (0,1)$. Consider a ball $B_R(0)$ in $\mathbb{R}^n$ and let $B^+_R = B_R(0) \cap \{x : x_n \geq 0\}$, using coordinates $x = (x_1, x_2, \dots, x_n)$ on $\mathbb{R}^n$. Suppose $u \in C^{2,\mu}(\overline{B^+_R})$ solves*

$$\begin{aligned} Lu = a^{ij} D_{ij} u + b^i D_i u + cu = f \text{ in } B^+_R, \\ u = \varphi \text{ on } S_R. \end{aligned}$$

where $a^{ij}, b^i, c : C^{0,\mu}(\overline{B^+_R})$ are coefficients, $f \in C^{0,\mu}(\overline{B^+_R})$, and $\varphi \in C^{2,\mu}(S_R)$. Assume the coefficients satisfy the bounds

$$\lambda |\xi|^2 \leq a^{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2 \text{ for } x \in B^+_R,\ \xi \in \mathbb{R}^n,$$

$$\sum_{i,j=1}^n \|a^{ij}\|'_{C^{0,\mu}(B^+_R)} + \sum_{i=1}^n R \|b^i\|'_{C^{0,\mu}(B^+_R)} + R^2 \|c\|'_{C^{0,\mu}(B^+_R)} \leq \beta,$$

for some constants $\lambda, \Lambda, \beta \in (0,\infty)$ such that $0 < \lambda \leq \Lambda$ and $f \in C^{0,\mu}(\overline{B^+_R})$. Then

$$|u|'_{2,\mu;B^+_{R/2}} \leq C \left(|u|_{0;B^+_R} + R^2 |f|'_{0,\mu;B^+_R} + |\varphi|'_{2,\mu;S_R} \right)$$

for some constant $C = C(n, \mu, \lambda, \Lambda, \beta) \in (0,\infty)$.

Proof. It suffices to prove the Schauder estimate in the special case that $\varphi \equiv 0$. To see this, recall that we can extend φ to a $C^{2,\mu}$ function on B^+_R by letting $\varphi(x', x_n) = \varphi(x')$. Let $v = u - \varphi$. Then

$$\begin{aligned} Lv = f - L\varphi \text{ in } B^+_R, \\ v = 0 \text{ on } S_R. \end{aligned}$$

Since $v = 0$ on S_R,

$$\begin{aligned} |u|'_{2,\mu;B^+_{R/2}} &\leq |\varphi|'_{2,\mu;B^+_{R/2}} + |v|'_{2,\mu;B^+_{R/2}} \\ &\leq |\varphi|'_{2,\mu;B^+_{R/2}} + C \left(|v|_{0;B^+_R} + R^2 |f|'_{0,\mu;B^+_R} + R^2 |L\varphi|'_{0,\mu;B^+_R} \right) \\ &\leq |\varphi|'_{2,\mu;B^+_{R/2}} + C \left(|u|_{0;B^+_R} + |\varphi|_{0;B^+_R} + R^2 |f|'_{0,\mu;B^+_R} + |\varphi|'_{2,\mu;B^+_R} \right) \\ &\leq C \left(|u|_{0;B^+_R} + R^2 |f|'_{0,\mu;B^+_R} + |\varphi|'_{2,\mu;S_R} \right), \end{aligned}$$

where $C = C(n, \mu, \lambda, \Lambda, \beta) \in (0, \infty)$.

To prove the Schauder estimate in the special case that $\varphi \equiv 0$, we can do the argument from scaling from the last lecture. Recall that we used interpolation and the simple abstract lemma to reduce to proof of the Schauder estimate to proving that for every $\delta > 0$,

$$R^{2+\mu}[D^2 u]_{2,\mu;B^+_{R/2}} \leq \delta R^{2+\mu}[D^2 u]_{2,\mu;B^+_R} + C\left(|u|'_{2;B^+_R} + R^2|f|'_{0,\mu;B^+_R}\right)$$

for some constant $C = C(n, \mu, \lambda, \Lambda, \beta, \delta) \in (0, \infty)$. We can apply interpolation and the simple abstract lemma here by, arguing as in the Hölder continuity lectures, extending u to a $C^{2,\mu}$ function on $B_R(0)$ such that

$$|u|'_{2;B^+_\rho} \leq |u|'_{2;B_\rho} \leq C(n)|u|'_{2;B^+_\rho},$$
$$[D^2 u]_{\mu;B^+_\rho} \leq [D^2 u]_{\mu;B_\rho} \leq C(n,\mu)[D^2 u]_{\mu;B^+_\rho}$$

and applying the interpolation inequalities and simple abstract lemma to u on $B_{R/2}(0)$ and $B_R(0)$.

Now take a sequence $u_k \in C^{2,\mu}(\overline{B^+_1})$ such that

$$L_k u_k = f_k \text{ in } B^+_1,$$
$$u_k = 0 \text{ on } S_1.$$

for some elliptic operator L_k with uniform ellipticity and $C^{0,\mu}$ bounds on the coefficients and $f_k \in C^{0,\mu}(\overline{B^+_1})$ and

$$[D^2 u_k]_{\mu;B^+_{1/2}} > \delta[D^2 u_k]_{\mu;B^+_1} + k\left(|u_k|_{2;B^+_1} + |f_k|_{0,\mu;B^+_1}\right).$$

Let $x_k, y_k \in B^+_{1/2}$ such that

$$\frac{|D^2 u_k(x_k) - D^2 u_k(y_k)|}{|x_k - y_k|^\mu} \geq \frac{1}{2}[D^2 u_k]_{\mu;B^+_{1/2}}$$

and $\rho_k = |x_k - y_k|$. Recall that $\rho_k \to 0$ as $k \to \infty$. After passing to a subsequence, it suffices to consider the following two cases:

(a) $\operatorname{dist}(x_k, S_1)/\rho_k \to \infty$ as $k \to \infty$

(b) $\gamma = \sup_k \operatorname{dist}(x_k, S_1)/\rho_k < \infty$.

In case (a), we can rescale as before letting

$$\widetilde{u}_k(x) = \frac{u_k(x_k + \rho_k x) - u_k(x_k) - \rho_k \sum_{i=1}^n D_i u_k(x_k) x_i - (1/2)\rho_k^2 \sum_{i,j=1}^n D_{ij} u_k(x_k) x_i x_j}{\rho_k^{2+\mu}[D^2 u_k]_{\mu;B^+_1}}$$

and letting $\widetilde{u}_k$ converges to some $\widetilde{u}$ in C^2 on compact subsets of $\mathbb{R}^n$. $\widetilde{u}$ will be a solution to a constant coefficient elliptic equation of the form $\widetilde{a}^{ij} D_{ij}\widetilde{u} = 0$ on $\mathbb{R}^n$, $D^2\widetilde{u}$ is nonconstant, and $[D^2\widetilde{u}]_{\mu;\mathbb{R}^n} < \infty$, contradicting the Liousville lemma.

For case (b), we instead take $z_k \in S_1$ such that $|x_k - z_k| \leq 2\gamma\rho_k$ and rescale letting

$$\widetilde{u}_k(x) = \frac{u_k(z_k + \rho_k x) - u_k(z_k) - \rho_k \sum_{i=1}^n D_i u_k(z_k) x_i - (1/2)\rho_k^2 \sum_{i,j=1}^n D_{ij} u_k(z_k) x_i x_j}{\rho_k^{2+\mu}[D^2 u_k]_{\mu;B^+_1}}$$

so that the upper-half-space remains fixed under scaling and $(x_k - z_k)/\rho_k$ and $(y_k - z_k)/\rho_k$ remain in a compact subset of $\mathbb{R}^n_+$. $\widetilde{u}_k$ will then converge to some $\widetilde{u}$ in C^2 on compact subsets of the upper-half-space $\mathbb{R}^n_+$ and $\widetilde{u}$ will satisfy to a constant coefficient elliptic equation of the form $\widetilde{a}^{ij} D_{ij} \widetilde{u} = 0$ on $\mathbb{R}^n_+$, $\widetilde{u} = 0$ on S, $D^2 \widetilde{u}$ is nonconstant on $\mathbb{R}^n_+$, and $[D^2 \widetilde{u}]_{\mu;\mathbb{R}^n_+} < \infty$.

We can rotate so that

$$\sum_{i,j=1}^n \widetilde{a}^{ij} D_{ij} \widetilde{u} = \sum_{i,j=1}^n \lambda_i D_{ii} \widetilde{u} = 0$$

however, in the process we rotate the domain of $\widetilde{u}$ to a half-space lying on one side of an $(n-1)$-dimensional linear subspace of $\mathbb{R}^n$. Let

$$w(x_1, x_2, \ldots, x_n) = \widetilde{u}\left(\sqrt{\lambda_1} x_1, \sqrt{\lambda_2} x_2, \ldots, \sqrt{\lambda_n} x_n\right)$$

so that w is harmonic on its domain, which is a half-space lying on one side of an $(n-1)$-dimensional linear subspace of $\mathbb{R}^n$. Rotate again so that the domain of w is $\mathbb{R}^n_+$. Now w is a harmonic function on $\mathbb{R}^n_+$ such that $w = 0$ on S, $D^2 w$ is nonconstant on $\mathbb{R}^n_+$, and $[D^2 w]_{\mu;\mathbb{R}^n_+} < \infty$.

Now extend w to a function on $\mathbb{R}^n$ by odd reflection, letting

$$w(x', x_n) = -w(-x', x_n) \text{ for } x' < 0,\ x_n \in \mathbb{R}.$$

Clearly w is continuous on $\mathbb{R}^n$ with $w = 0$ on S and Dw is defined and continuous on $\mathbb{R}^n$ with $Dw(0, x_n)$ unambiguously defined on S. Since $w = 0$ on S and w is harmonic,

$$D_{nn} w = \Delta w - \sum_{i=1}^{n-1} D_{ii} w = 0 \text{ on } S.$$

Thus, using $w = 0$ on S, $D^2 w$ exists and is continuous on S with $D^2 w = 0$ on S. Now we extended w to a harmonic function on $\mathbb{R}^n$ such that $D^2 w$ is nonconstant on $\mathbb{R}^n_+$ and $[D^2 w]_{\mu;\mathbb{R}^n_+} < \infty$, contradicting the Liousville lemma. □

Now let $\mu \in (0,1)$ and Ω be a $C^{2,\mu}$ domain in $\mathbb{R}^n$. Suppose that $u \in C^{2,\mu}(\overline{\Omega})$ solves

$$\begin{aligned} Lu = a^{ij} D_{ij} u + b^i D_i u + cu &= f \text{ in } B_R^+, \\ u &= \varphi \text{ on } S_R. \end{aligned}$$

where $a^{ij}, b^i, c : C^{0,\mu}(\overline{\Omega})$ are coefficients satisfying

$$\lambda |\xi|^2 \leq a^{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2 \text{ for } x \in B_R^+,\ \xi \in \mathbb{R}^n,$$
$$\sum_{i,j=1}^n \|a^{ij}\|'_{C^{0,\mu}(B_R^+)} + \sum_{i=1}^n R \|b^i\|'_{C^{0,\mu}(B_R^+)} + R^2 \|c\|'_{C^{0,\mu}(B_R^+)} \leq \beta,$$

for some constants $\lambda, \Lambda, \beta \in (0, \infty)$ and $f \in C^{0,\mu}(\overline{\Omega})$ and $\varphi \in C^{2,\mu}(\partial\Omega)$. Given a point $y \in \partial\Omega$, there exists a small ball $B_R(y)$ and a $C^{2,\mu}$ diffeomorphism $\Psi : B_R(y) \to \mathbb{R}^n$ such that

$$\Psi(\Omega) \cap B_R(0) = B_R^+, \quad \Psi(\partial\Omega) \cap B_R(0) = S_R$$

and Ψ is close to the identity map in $C^{2,\mu}$. (Note of course that the domain Ω could be bad enough behaved that we will have to choose a very small R.) Moreover, we can extend φ to a $C^{2,\mu}$

function on $\overline{\Omega} \cap B_R(y)$. Let $\widetilde{u} = u \circ \Psi^{-1}$, $\widetilde{a}^{ij} = a^{ij} \circ \Psi^{-1}$, $\widetilde{b}^i = b^i \circ \Psi^{-1}$, $\widetilde{c} = c \circ \Psi^{-1}$, $\widetilde{f} = f \circ \Psi^{-1}$, and $\widetilde{\varphi} = \varphi \circ \Psi^{-1}$ on $\overline{B_R^+}$. Then

$$\begin{aligned} D_i u &= D_i(\widetilde{u} \circ \Psi) = D_k \widetilde{u} D_i \Psi^k, \\ D_{ij} u &= D_j(D_k \widetilde{u} D_i \Psi^k) = D_{kl} \widetilde{u} D_i \Psi^k D_j \Psi^l + D_k \widetilde{u} D_{ij} \Psi^k, \end{aligned}$$

and thus

$$\widetilde{a}^{ij} D_i \Psi^k D_j \Psi^l D_{kl} \widetilde{u} + \widetilde{a}^{ij} D_{ij} \Psi^k D_k \widetilde{u} + \widetilde{b}^i D_i \Psi^k D_k \widetilde{u} + \widetilde{c}\widetilde{u} = \widetilde{f} \text{ in } B_R^+.$$

Moreover, $\widetilde{u} = \widetilde{\varphi}$ on S_R. Therefore, by the Schauder estimates above,

$$|\widetilde{u}|'_{2,\mu;B^+_{R/2}} \leq C \left(|\widetilde{u}|_{0;B^+_R} + R^2 |\widetilde{f}|'_{0,\mu;B^+_R} + |\widetilde{\varphi}|'_{2,\mu;S_R} \right)$$

for some constant $C = C(n, \mu, \lambda, \Lambda, \beta) \in (0, \infty)$, implying that

$$|u|'_{2,\mu;\Omega \cap B_{R/4}(y)} \leq C \left(|u|_{0;\Omega \cap B_R(0)} + R^2 |f|'_{0,\mu;\Omega \cap B_R(0)} + |\varphi|'_{2,\mu;\partial\Omega \cap B_R(0)} \right)$$

for $C = C(n, \mu, \lambda, \Lambda, \beta) \in (0, \infty)$.

Theorem 1 (Global Schauder estimates)**.** *Let $\mu \in (0,1)$ and Ω be a bounded $C^{2,\mu}$ domain. Suppose $u \in C^{2,\mu}(\overline{\Omega})$ solves the uniformly elliptic equation*

$$Lu = a^{ij} D_{ij} u + b^i D_i u + cu = f \text{ in } \Omega,$$

where the coefficients $a^{ij}, b^i, c \in C^{0,\mu}(\overline{\Omega})$ satisfy

$$\begin{aligned} &\lambda |\xi|^2 \leq a^{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2 \text{ for } x \in \Omega,\ \xi \in \mathbb{R}^n, \\ &|a^{ij}|_{0,\mu;\Omega} + |b^i|_{0,\mu;\Omega} + |c|_{0,\mu;\Omega} \leq \beta, \end{aligned}$$

for some constants $\lambda, \Lambda, \beta > 0$ and $f \in C^{0,\mu}(\overline{\Omega})$. Then

$$|u|_{2,\mu;\Omega} \leq C \left(|u|_{0;\Omega} + |f|_{0,\mu;\Omega} + |\varphi|_{2,\mu;\partial\Omega} \right)$$

for some constant $C = C(n, \mu, \lambda, \Lambda, \beta, \Omega) \in (0, \infty)$. In particular, if $c \leq 0$ in Ω, then by the weak maximum principle,

$$|u|_{2,\mu;\Omega} \leq C \left(|f|_{0,\mu;\Omega} + |\varphi|_{2,\mu;\partial\Omega} \right)$$

for some constant $C = C(n, \mu, \lambda, \Lambda, \beta, \Omega) \in (0, \infty)$.

Proof. Recall that for every $y \in \partial\Omega$ there exists $R = R(y) > 0$ such that

$$|u|'_{2,\mu;\Omega \cap B_{R/4}(y)} \leq C \left(|u|_{0;\Omega \cap B_R(y)} + R^2 |f|'_{0,\mu;\Omega \cap B_R(y)} + |\varphi|'_{2,\mu;\partial\Omega \cap B_R(y)} \right) \tag{1}$$

for $C = C(n, \mu, \lambda, \Lambda, \beta) \in (0, \infty)$. Since Ω is a bounded, we can cover $\partial\Omega$ by a finite collection of balls $B_{R(y_j)/16}(y_j)$, $j = 1, 2, \ldots, N$, where N depends on Ω. Let $R_j = R(y_j)$ for each j, $R_* = \inf_j R_j$, and

$$\Omega' = \{x \in \Omega : \text{dist}(x, \partial\Omega) < R_*/16\}.$$

Observe that $\overline{\Omega}$ is covered by Ω' and $B_{R_j/8}(y_j)$: if $x \in \overline{\Omega} \setminus \Omega'$, there exists $z \in \partial\Omega$ such that $|x - z| < R_*/16$ and there exists j such that $z \in B_{R_j/16}(y_j)$, so $x \in B_{R_j/8}(y_j)$. Recall that

$$|u|_{2,\mu;\Omega'} \leq C \left(|u|_{0;\Omega} + |f|_{0,\mu;\Omega} \right) \tag{2}$$

for $\Omega' \subset\subset \Omega$ for some constant $C = C(n, \mu, \lambda, \Lambda, \beta, \Omega', \Omega) \in (0, \infty)$.

Now we can obtain supremum estimates on u, Du and D^2u by combining (1) and (2). We can also bound $[D^2u]_{\mu;\Omega}$ by combining (1), (2), and the supremum estimate on D^2u. Note that it is important that Ω' and the balls $B_{R_j/8}(y_j)$ cover $\overline{\Omega}$ and we have Schauder estimates bounding u on Ω' and balls $B_{R_j/4}(y_j)$, that way whenever $x, y \in \overline{\Omega}$ with $|x - y| < R_*/8$, either both x, y are in Ω' or both x, y in $B_{R_j/4}(y_j)$ for some j.

To be specific, let $x \in \overline{\Omega}$. If $x \in \Omega'$, then

$$|u(x)| + |Du(x)| + |D^2u(x)| \leq C \left(|u|_{0;\Omega} + |f|_{0,\mu;\Omega}\right)$$

by (2). If instead $x \in B_{R_j/8}(y_j)$ for some j, then

$$|u(x)| + R_j|Du(x)| + R_j^2|D^2u(x)| \leq C \left(|u|_{0;\Omega\cap B_{R_j}(y_j)} + R_j^2|f|'_{0,\mu;\Omega\cap B_{R_j}(y_j)} + |\varphi|'_{2,\mu;\partial\Omega\cap B_{R_j}(y_j)}\right)$$

by (1). Therefore,

$$|u(x)| + |Du(x)| + |D^2u(x)| \leq C \left(|u|_{0;\Omega} + |f|'_{0,\mu;\Omega} + |\varphi|'_{2,\mu;\partial\Omega}\right) \tag{3}$$

for some constant $C = C(n, \mu, \lambda, \Lambda, \beta, \Omega', \Omega) \in (0, \infty)$.

Let $x, y \in \overline{\Omega}$. If $x, y \in \Omega'$, then

$$\frac{|D^2u(x) - D^2u(y)|}{|x - y|^\mu} \leq [D^2u]_{\mu;\Omega'} \leq C \left(|u|_{0;\Omega} + |f|'_{0,\mu;\Omega}\right)$$

by (2). If instead $x \in \overline{\Omega} \setminus \Omega'$ and $|x - y| < R_*/8$, then $x \in B_{R_j/8}(y_j)$ for some j and $y \in B_{R_j/4}(y_j)$ so

$$\frac{|D^2u(x) - D^2u(y)|}{|x - y|^\mu} \leq [D^2u]_{\mu;B_{R_j/4}(y_j)} \leq CR_j^{-\mu} \left(|u|_{0;\Omega\cap B_{R_j}(y_j)} + R_j^2|f|'_{0,\mu;\Omega\cap B_{R_j}(y_j)} + |\varphi|'_{2,\mu;\partial\Omega\cap B_{R_j}(y_j)}\right)$$

by (1). If instead $x \in \overline{\Omega} \setminus \Omega'$ and $|x - y| \geq R_*/8$, then

$$\frac{|D^2u(x) - D^2u(y)|}{|x - y|^\mu} \leq \frac{2}{(R_*/8)^\mu} \sup_\Omega |D^2u| \leq CR_*^{-\mu} \left(|u|_{0;\Omega} + |f|'_{0,\mu;\Omega} + |\varphi|'_{2,\mu;\partial\Omega}\right)$$

by (3). Therefore,

$$\frac{|D^2u(x) - D^2u(y)|}{|x - y|^\mu} \leq C \left(|u|_{0;\Omega} + |f|'_{0,\mu;\Omega} + |\varphi|'_{2,\mu;\partial\Omega}\right)$$

for some constant $C = C(n, \mu, \lambda, \Lambda, \beta, \Omega', \Omega) \in (0, \infty)$. □